

## 1.5 Solutions Sets of Linear Systems

**Homogeneous System:**

$$A\mathbf{x} = \mathbf{0}$$

( $A$  is  $m \times n$  and  $\mathbf{0}$  is the zero vector in  $\mathbf{R}^m$ )

**EXAMPLE:**

$$\begin{aligned}x_1 + 10x_2 &= 0 \\2x_1 + 20x_2 &= 0\end{aligned}$$

Corresponding matrix equation  $A\mathbf{x} = \mathbf{0}$ :

$$\begin{bmatrix} 1 & 10 \\ 2 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

**Trivial solution:**

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \mathbf{x} = \mathbf{0}$$

The homogeneous system  $A\mathbf{x} = \mathbf{0}$  **always** has the **trivial solution**,  $\mathbf{x} = \mathbf{0}$ .

Nonzero vector solutions are called **nontrivial solutions**.

Do **nontrivial** solutions exist?

$$\begin{bmatrix} 1 & 10 & 0 \\ 2 & 20 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Consistent system with a free variable has infinitely many solutions.

A homogeneous equation  $A\mathbf{x} = \mathbf{0}$  has nontrivial solutions if and only if the system of equations has \_\_\_\_\_.

**EXAMPLE:** Determine if the following homogeneous system has nontrivial solutions and then describe the solution set.

$$2x_1 + 4x_2 - 6x_3 = 0$$

$$4x_1 + 8x_2 - 10x_3 = 0$$

*Solution:*

There is at least one free variable (why?)

$\Rightarrow$  nontrivial solutions exist

$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 4 & 8 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

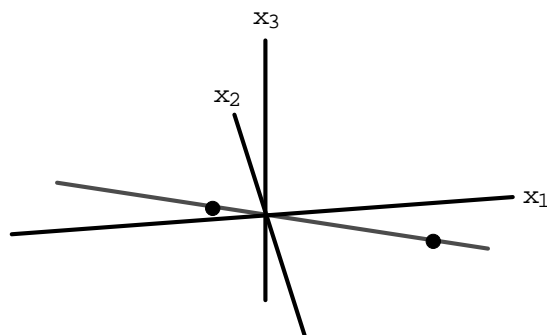
$$x_1 =$$

$x_2$  is free

$$x_3 =$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = \frac{x_2}{1} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2 \mathbf{v}$$

Graphical representation:



solution set =  $\text{span}\{\mathbf{v}\}$  = line through  $\mathbf{0}$  in  $\mathbf{R}^3$

**EXAMPLE:** Describe the solution set of

$$2x_1 + 4x_2 - 6x_3 = 0$$

$$4x_1 + 8x_2 - 10x_3 = 4$$

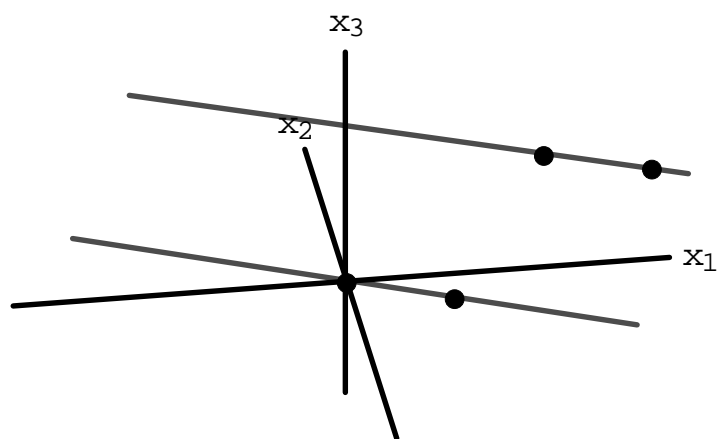
( same left side as in the previous example)

*Solution:*

$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 4 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{p} + x_2 \mathbf{v}$$



Parallel solution sets of  $A\mathbf{x} = \mathbf{0}$  &  $A\mathbf{x} = \mathbf{b}$

## Recap of Previous Two Examples

Solution of  $A\mathbf{x} = \mathbf{0}$

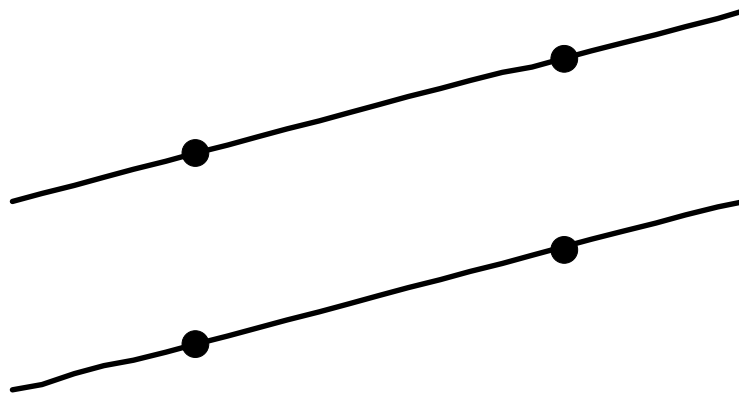
$$\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2 \mathbf{v}$$

$\mathbf{x} = x_2 \mathbf{v}$  = parametric equation of line passing through  $\mathbf{0}$  and  $\mathbf{v}$

Solution of  $A\mathbf{x} = \mathbf{b}$

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{p} + x_2 \mathbf{v}$$

$\mathbf{x} = \mathbf{p} + x_2 \mathbf{v}$  = parametric equation of line passing through  $\mathbf{p}$  parallel to  $\mathbf{v}$



Parallel solution sets of  
 $A\mathbf{x} = \mathbf{b}$  and  $A\mathbf{x} = \mathbf{0}$

### THEOREM 6

Suppose the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for some given  $\mathbf{b}$ , and let  $\mathbf{p}$  be a solution. Then the solution set of  $A\mathbf{x} = \mathbf{b}$  is the set of all vectors of the form  $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ , where  $\mathbf{v}_h$  is any solution of the homogeneous equation  $A\mathbf{x} = \mathbf{0}$ .

**EXAMPLE:** Describe the solution set of  $2x_1 - 4x_2 - 4x_3 = 0$ ; compare it to the solution set  $2x_1 - 4x_2 - 4x_3 = 6$ .

*Solution:* Corresponding augmented matrix to  $2x_1 - 4x_2 - 4x_3 = 0$ :

$$\begin{bmatrix} 2 & -4 & -4 & 0 \end{bmatrix} \sim \quad \text{(fill-in)}$$

Vector form of the solution:

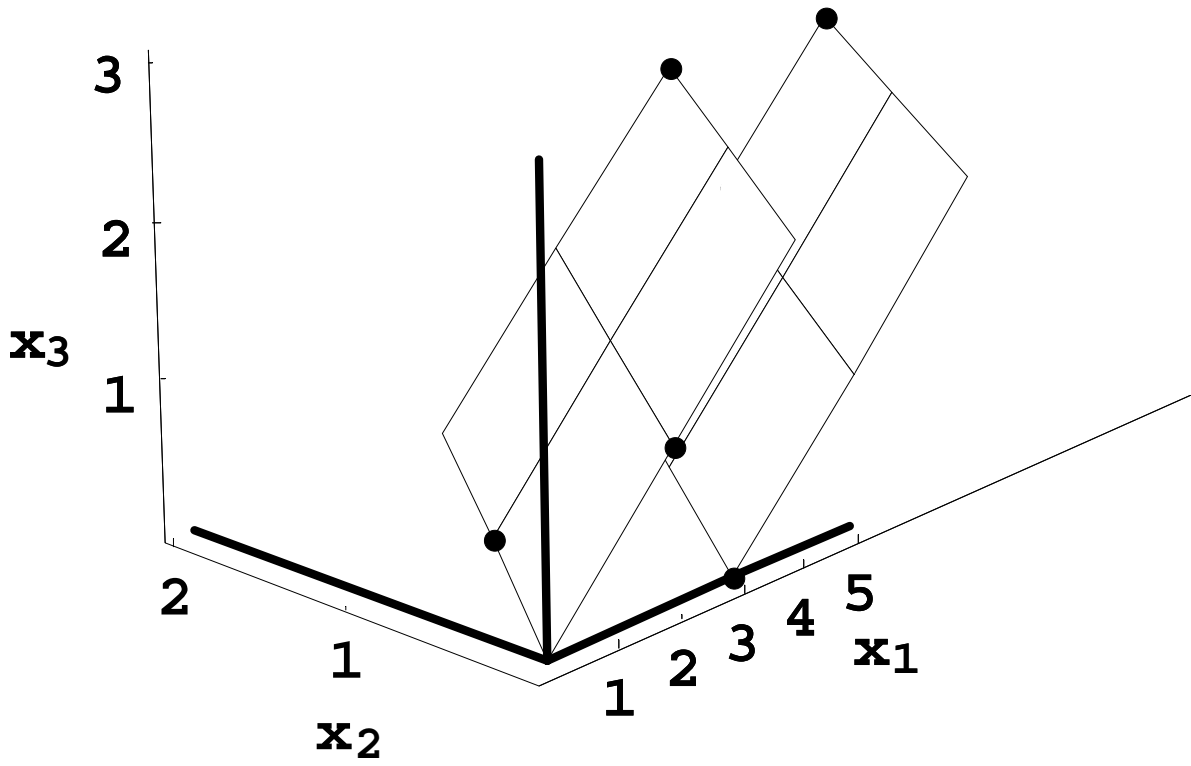
$$\mathbf{v} = \begin{bmatrix} 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \text{---} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \text{---} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Corresponding augmented matrix to  $2x_1 - 4x_2 - 4x_3 = 6$ :

$$\begin{bmatrix} 2 & -4 & -4 & 6 \end{bmatrix} \sim \quad \text{(fill -in)}$$

Vector form of the solution:

$$\mathbf{v} = \begin{bmatrix} 3 + 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} + \text{---} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \text{---} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$



Parallel Solution Sets of  $Ax = \mathbf{0}$  and  $Ax = \mathbf{b}$